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SOBOLEV SPACES AND SYMBOLIC CALCULUS

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ABSTRACT

We use elementary theory of distributions and geometry of Euclidean spaces to obtain a theorem on the symbolic calculus of several variables in spaces of Fourier series with weights.

1. Let f be a real function on R^N with continuous partial derivatives up to order p, and S a closed subset of R^N of dimension $< N$ (roughly speaking). Then it is sometimes possible to interpolate f, on S, by a function g in R^N, with more than p derivatives in $L^2(R^N)$, so that $\hat{g}$ exhibits better integrability than $\hat{f}$. In our main theorem we establish an interpolation of this kind; although it appears artificial, it is well adapted for a certain problem in Banach algebras. Following a suggestion of the referee, we treat this problem in a Corollary.

2. Let $\beta > 0$ and let S be a closed subset of R^N. We say that S has property S_β if, for each h in $(0, 1)$, S is contained in the union of at most $CH^{-\beta}$ open balls of radius h.

THEOREM. *Let f belong to $C^p(R^N)$, ($p = 0, 1, 2, \cdots$) and let S have property S_β. Let $q \geqq p + 1$ and $N - \beta > 2(q - p)$. Then there exists a function g on R^N, such that $g \mid S = f \mid S$, and g has generalized derivatives of order $0, 1, \cdots, q$ in $L^2(R^N)$.*

PROOF. Let f_0 be the restriction of f to S, and let g be the extension of f_0 to R^N, produced by Whitney's Theorem [2, VI]. Then $g \in C^p(R^N)$ and, for every partial derivative D^q of order $q > p$, $|D^q g| = O(d(x, S))^{p-q}$ near S. Since g is C^∞ off S, we can achieve this with a function of compact support. Then $D^q g$ is defined almost everywhere and for large Y

$$m\{|D^q G| > Y\} \leqq m\{d(x, S) < CY^{-1/q-p}\} = O(Y^{-\eta}),$$

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where $\eta = (N-\beta)(q-p)^{-1}$. Now $D^qG \in L^2(R^N)$ provided $\eta > 2$, but we still have to verify that D^qg is the derivative in the sense of distributions.

Let $\psi_k(x) = k^N\psi(kx)$ be an approximate identity, with $\psi_1 \in C^\infty$ and $\psi_1(x) = 0$ for $|x| > 1$. Writing $g_k = g * \psi_k$ we observe that

$$D^q g_k = O(d(x,S))^{p-q}, \quad \text{if } d(x,S) \geqq 2k^{-1},$$

$$D^q g_k = O(k^{q-p}), \qquad \text{always.}$$

Hence the first line is true unconditionally, and thus D^qg exists as a distribution in L^2 by dominated convergence. This ends the proof.

In view of the applications, we write the conclusion as an inequality on $\hat{g}$: $\int\int |\hat{g}(u)|^2(1+|u|)^{2q}du < +\infty$. By the Schwarz inequality we have

COROLLARY 1. *$\int\int |\hat{g}(u)|(1+|u|)^\gamma du < +\infty$ if $2\gamma < 2q - N$.*

3. Let now $\alpha > 0$ and let A_α be the Banach algebra of functions $F(x) = \Sigma\, a_n e^{inx}$, with finite norm $\|F\|_\alpha \equiv \Sigma(1+|n|)^\alpha |a_n|$. We remark that if $\alpha \geqq 1$, then F is of class $C^1(R)$, while F is Hölder-continuous of order α when $0 < \alpha < 1$. For real functions F in A_α,

$$\|e^{iF}\|_\alpha \leqq C(1+\|F\|_\alpha)^\tau;$$

$\tau > 0$ and C depend only on α. For details on the best value of τ, see [1].

COROLLARY 2. *There is an integer $p(\alpha)$ depending only on α, with this property: whenever $F_1, \cdots, F_N$ are real and belong to A_α, and F belongs to $C^p(R^N)$, then $f(F_1, \cdots, F_N)$ is A_α.*

We haven't given the details, but the calculations below can be sharpened to yield

$$p(\alpha) \leqq \alpha^{-1} + O(1) \quad \text{for } 0 < \alpha \leqq 1,$$

$$p(\alpha) \leqq \alpha/2 + O(1) \quad \text{for } \alpha \geqq 1.$$

A small advantage can be gained by allowing fractional numbers p.

To prove Corollary 2 we let S be the graph $\{(F_1(t), \cdots, F_N(t)) : 0 \leqq t \leqq 2\pi\}$. Then S has property S_β, with $\beta = \max(1, \alpha^{-1})$. We apply Corollary 1, and the cited theorem of Leblanc, with $\gamma = \tau(\alpha)$. (We can always assume that $N > \beta$.) This succeeds if there is an integer $q = q(N, \alpha)$ so that

(i) $N - \beta + 2p > 2q$,

(ii) $2q > 2\tau + N$.

Now (i) and (ii) certainly admit a solution q if $N-\beta+2p>2\tau+N+2$, or $2p>2\tau+\beta+2$.

References

1. N. Leblanc, *Les fonctions qui opèrent dans certains algèbres à poids*, Math. Scand. **25** (1969), 190–194.

2. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

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